

Tutorial 6

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① Show that $\sum_{k=1}^{\infty} \frac{1}{k^x}$ converges

- Ⓐ Pointwise on $(1, \infty)$
- Ⓑ Uniformly on $[1+\delta, \infty) \quad \forall \delta > 0$
- Ⓒ Does not converge uniformly on $(1, \infty)$
- Ⓓ The series describes a continuous fn $\mathcal{T}: (1, \infty) \rightarrow \mathbb{R}$

Soln: $f_n \rightarrow f$ pointwise on $A \Leftrightarrow \forall x \in A, \forall \varepsilon > 0 \exists N$ s.th. $\forall n,$

$$n > N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon$$

We are asking whether $f_n(x) = \sum_{k=1}^n \frac{1}{k^x}$ converges pointwise

$$\text{to } \sum_{k=1}^{\infty} \frac{1}{k^x}.$$

Fix $x \in (1, \infty)$. Recall $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$

So since $x > 1$, $\sum_{k=1}^{\infty} \frac{1}{k^x}$ converges. Now fix $\varepsilon > 0$,

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \sum_{k=n+1}^{\infty} \frac{1}{k^x} \right| \\ &\leq \left| \int_n^{\infty} t^{-x} dt \right| \quad \text{b.e. } \frac{1}{k^x} \text{ is monotone decreasing and positive.} \end{aligned}$$

$$= \left| \frac{-n^{-x+1}}{1-x} \right| \quad \text{let } x = 1+\delta, \delta > 0$$

$$= \frac{1}{\delta} n^{-\delta}$$

$< \varepsilon$ pick n large.

\leftarrow \subset prior \cap \cdots γ

So $\sum_{k=1}^n \frac{1}{k^x}$ converges pointwise to $\sum_{k=1}^{\infty} \frac{1}{k^x}$ on $(1, \infty)$

(b) Def: $f_n \rightarrow f$ uniformly $\Leftrightarrow \forall \varepsilon > 0, \exists N, \forall x$

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Fix $\delta > 0$. We will use the M-test to show

The series converges uniformly on $[1+\delta, \infty)$

Recall: (Weierstrass M-test) let $\sum_{k=1}^{\infty} f_k$ be a series.

If $\forall k, \exists M_k \quad \forall x \in A \quad |f_k(x)| \leq M_k$ and $\sum_{k=1}^{\infty} M_k < \infty$

then $\sum_{k=1}^{\infty} f_k$ converges uniformly on A.

So we need to bound $\frac{1}{k^x}$ on $[1+\delta, \infty)$.

$$\left| \frac{1}{k^x} \right| \leq \frac{1}{k^{1+\delta}} := M_k$$

And notice $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^{1+\delta}} < \infty$ when $\underline{\delta > 0}$

By the M-criterion, the series converges uniformly on $[1+\delta, \infty)$.

(c) We need to show $\exists \varepsilon > 0$ s.t. $\forall N \quad \exists x \in (1, \infty)$, $\exists n$ s.t.

$$n > N \quad \text{and} \quad \left| \sum_{k=1}^{\infty} \frac{1}{k^x} - \sum_{k=1}^n \frac{1}{k^x} \right| \geq \varepsilon.$$

Try $\varepsilon = 1/2$ and fix N . Notice we have an inequality

$$\begin{aligned}
 \left| \sum_{k=N+1}^{\infty} \frac{1}{k^x} \right| &\geq \left| \int_{N+1}^{\infty} t^{-x} dt \right| \\
 &= \left| \frac{(N+1)^{-x+1}}{1-x} \right| \quad \text{let } x = 1+\delta \\
 &= \frac{1}{\delta} (N+1)^{-\delta} \quad \text{Choose } \delta = \frac{1}{N+1} \\
 &= (N+1)^{-\frac{1}{N+1}}
 \end{aligned}$$

Notice $(N+1)^{-\frac{1}{N+1}} \rightarrow 1$ as $N \rightarrow \infty$ so choose $n > N$ s.t.

$$n^{-1/n} \geq 1/2$$

Then for $x = 1 + \frac{1}{N+1}$ and the chosen n , we have

$$\left| \sum_{k=n}^{\infty} \frac{1}{k^x} \right| \geq 1/2$$

which finishes the proof that the series does not converge uniformly on $(1, \infty)$.

d) Show that $\sum_{k=1}^{\infty} \frac{1}{k^x}$ is continuous $\forall x \in (1, \infty)$

Proof: Fix $x \in (1, \infty)$. Continuity is local, so we only need to show f is continuous on an open set containing x .

Problem 2 In the last example, the pointwise limit happened to be a continuous function. This does not happen in general.

Here is an example of a sequence of functions which converges pointwise but not uniformly, and whose limit will not be continuous.

let $f_n : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$, $f_n(x) = (\cos x)^n$

Prop: $f_n \rightarrow f$ pointwise, where $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$

Proof: Notice that $|\cos x| \leq 1$, with $|\cos x| = 1 \Leftrightarrow x \neq 0 \quad x \in (-\pi/2, \pi/2)$

, if $x \neq 0$ then $|(\cos x)^n| \leq \delta^n$ for some $0 \leq \delta < 1$

Since $\delta < 1$, $\delta^n \rightarrow 0$ as $n \rightarrow \infty$.

. if $x = 0$ then $(\cos 0)^n = 1 \quad \forall n$, so $|\cos(0)^n - 1| = 0 < \varepsilon$.



